COMP4019 - Lab Session 9 – Mockup Exam

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1 Amortized Analysis

1.1 Potential Functions

Note: This is one is just a warm-up on the potential method and an excuse to think about trees.
Consider a tree data structure t. We note:

- r the root of t
- $n(x)$ the number of nodes in the subtree rooted at x
- $h(x)$ the height of a node x
- d_{low} and d_{high}, the minimum and maximum depths of leaves

Among the following options, mark which potential functions are valid *(note that since you do not know the structure of t or the algorithm performed on it, you cannot know which ones if any make sense to perform amortized analysis, only whether they are valid or not)*. Argue why/why not. Determine the potential of an empty tree for each.

1. $\phi_1(t) = n(r)^2$
2. $\phi_2(t) = d_{\text{low}} - d_{\text{high}}$
3. $\phi_3(t) = h(r) - d_{\text{low}}$
4. $\phi_4(t) = 2^{h(r)} - n(r)$

1.2 Move To Front

Note: This is not an easy application of amortized complexity, but I am running out of "simple/self-contained" examples to illustrate the concept. It is an interesting and different application of it nonetheless, and one in which the aggregate method does not work.
Linked lists are notorious for having bad complexity for *random* accesses. To address this shortcoming, one approach is that of *self-organizing lists*. The idea of self-organizing lists is that the nodes in the list are dynamically rearranged based on how frequently they are accessed (it is thus in some way related to the concept of *consolidation*).

Move to front (MTF) is a heuristic technique for self-organizing lists that consists in moving an element to the front of the list when it is accessed. For instance, accessing D in the following list has the indicated effect:

$$(A, B, C, \underline{D}, E, F) \rightarrow (\underline{D}, A, B, C, E, F)$$

The complexity of these accesses can be compared to that of an "idealized" heuristic. Consider a heuristic IDL that knows in advance in what order elements will be accessed, and has the ability to reorder the list once (for free) before starting these accesses. The IDL heuristic does not reorder the list after each `find` operation (although it can further be shown that this does not change the result below).

Show that the *amortized* cost of $\texttt{find}_{\mathrm{MTF}}$ in a singly-linked list is within a constant multiplicative factor of that of $\texttt{find}_{\mathrm{IDL}}$.

1. Start by finding a potential function that captures the difference in potential between a list ordered according to MTF and one according to IDL. Both heuristics are given an identical list (thus the potential must be 0), then IDL performs a free one-time re-ordering, then `find` operations may be performed and their cost analyzed.

2. Compute the change in your potential function after executing a `find` operation in MTF and in IDL.

3. Show the stated bound.

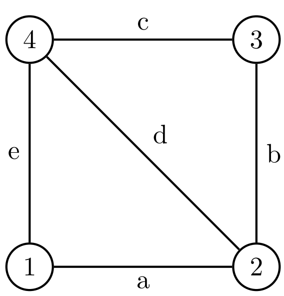
2 Graphs

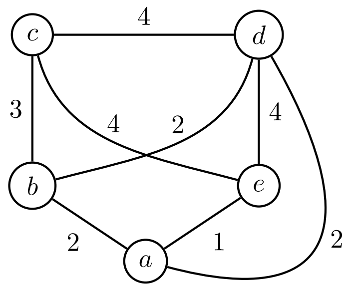
Note: Graph stuff. Good exercise to test your understanding on graphs, and manipulate formal definitions and statements on them.

The *edge-to-vertex dual* of an unidrected graph $G = (V, E)$ is an undirected graph $G^\# = (V^\#, E^\#)$ such that:

- $V^\# = E$;
- $E^\# = \{(e_i, e_j) \in E^2 \mid e_i \neq e_j \text{ and } \exists\, v \in V : v \in e_i \wedge v \in e_j\}$.

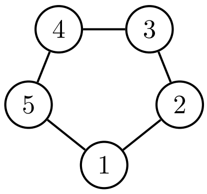
In other words: for each edge in G there is a vertex in $G^\#$; and for every two edges in G that share a vertex, there is an edge between their corresponding vertices in $G^\#$. Below is an example of a graph G (left) transformed into its edge-to-vertex dual $G^\#$ (right).





1. Write (in pseudocode) a "edge-to-vertex dual" conversion algorithm `e2v_convert`.
2. Show that if G is a *cycle graph* (a graph consisting of a single cycle – see example below), the edge-to-vertex dual conversion is *isomorphic* (that is, the resulting graph is identical, up to relabeling of the nodes and vertices).

Example of a cycle graph:



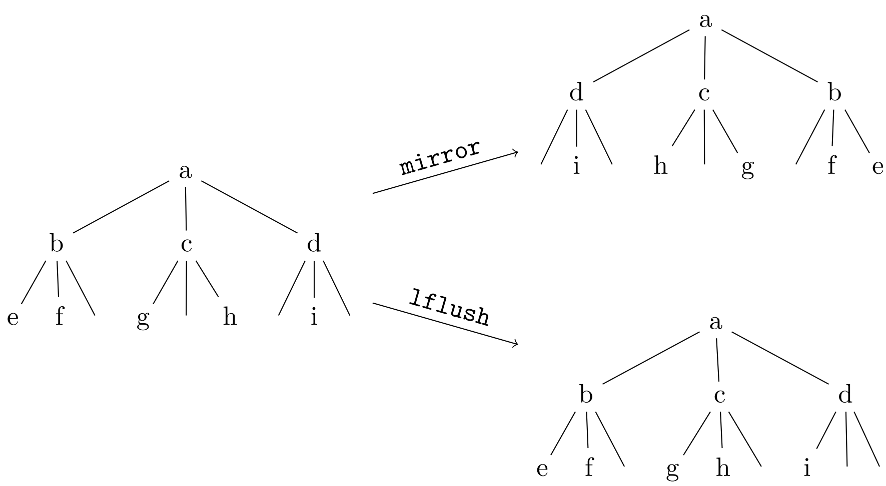
3 Trees

Note: Tests simple algorithms on trees and your ability to reason about transformations. Inspired by a Google coding interview question.

Consider a ternary tree t. Each node has `left`, `middle` and `right` children (some potentially empty).

1. Write (in pseudocode) the `mirror` operation that swaps the left and right children for each subtree of a tree t.
2. Write (in pseudocode) the `lflush` operation that flushes children towards the left, such that any empty child pointer is replaced by the pointer to its right.
3. Show that `lflush(mirror(`t`))` $=$ `lflush(mirror(lflush(`t`)))`. *Hint: consider each level independently.*

An example of the two operations in action (if a node has three empty children, the pointers are not shown):



4 Dynamic Programming

Note: Tests your understanding of the usability of dynamic programming.

For each of the following problem statements, determine whether (1) it demonstrates *optimal substructure* and whether (2) it possesses *overlapping subproblems*. Briefly explain why or why not for both.

1. Computing F_n, the n-th Fibonacci number recursively *(reminder: $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$)*;
2. Binary search for a given key in a sorted array;
3. The rod-cutting problem *(reminder: given a rod of length n and a table of prices $p_1, \ldots, p_n$ for pieces of lengths $1, \ldots, n$, cut the rod into pieces maximizing the total price)*;
4. Variant on the rod-cutting problem where each "cut" has a given fixed cost c;

5 Mysterious Algorithm

Note: Algorithm complexity analysis. Should be fairly straighforward. Be mindful about what is/are the "variable(s)" here.

Consider the following pseudocode of an algorithm on arrays:

function $\textsc{Mystery}(A, a, b, k)$
  if $a = b$ **then**
    return $A[a]$
  $c \leftarrow \textsc{Auxiliary}(A, a, b)$
  if $k = c$ **then**
    return $A[k]$
  else if $k < c$ **then**
    return $\textsc{Mystery}(A, a, c - 1, k)$
  else
    return $\textsc{Mystery}(A, c + 1, b, k)$

Random accesses are in $\Theta(1)$. The $\textsc{auxiliary}$ function runs in $O(b - a)$, and returns c such that $a \leq c < b$. Determine (and justify) the worst-case complexity of $\textsc{mystery}(A, 0, n - 1, k)$.