Binary Search Trees

Advanced Algorithms and Data Structures - Lecture 3A

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Dynamic Sets

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Dictionaries

In practice elements of a dynamic sets will be pairs:

A key used for searching, a value to be returned

Such a dynamic set is also called a dictionary

Example: In a database of students, the key could be the ID number, the value the name of the student (and all other relevant data)

$$\{(7, \text{Monica}), (5, \text{Richard}), (1, \text{Fang}), (10, \text{Wei}),$$
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For the study of the algorithms just the keys are relevant
I will describe the algorithms just using a set of keys
Exercise: Extend them to include the values

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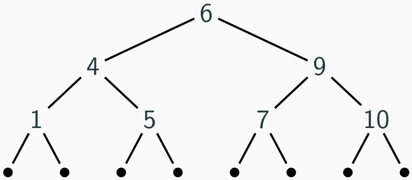
Can we find a data structure for which all three operations are efficient?

BST

Binary Search Trees



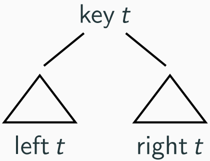
The operations of search, insert, delete
can be done in $O(k)$ time
where k is the depth of the tree

But tricky to keep k small: $k \sim O(\log n)$

There are more advanced variants that guarantee this: Red-Black Trees

The BST Property

For every tree t, we use the notation:



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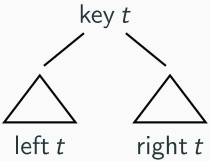

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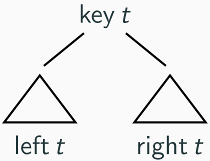

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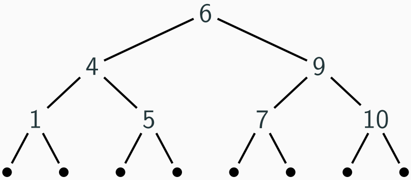

$4, 1, 5 < 6$
$9, 7, 10 > 6$
recursively:
$1 < 4, 5 > 4$
$7 < 9, 10 > 9$

Trees with key-value pairs

In practical applications:
nodes will contain pairs $\langle k, v \rangle$ of
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For the sake of the definition of the algorithms, we only use keys
Exercise: modify the algorithms with key-value pairs

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  containing elements from an ordered type Key
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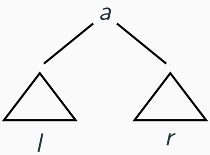

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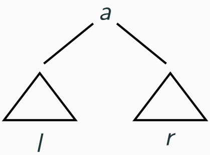

In functional programming it is defined as this inductive data type

```
data BST = Nil | Node Key BST BST
```

Component Functions

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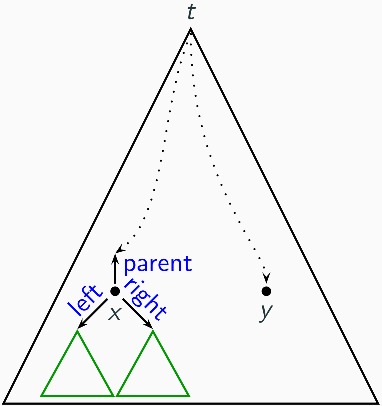
Imperative Implementation

In imperative programming, we use pointers to nodes, with methods giving the two children and the parent (See Ch.12 of IA)

(We can do it in functional programming too: using paths or defining them directly with those fields (Exercise))

Global Objects, Local Pointers

Work with a global tree t and with pointers x, y to subtrees/nodes



We can move around the tree with operations on pointers

- (parent x) the immediate precursor of x; Nil if x is the root
- (left x) and (right x) the children of x; Nil if they are leaves

Searching

Searching a tree t for a key k is done by following a single path
At each node x:

- If $k = \text{key}\, x$, we have found it!
- If $k < \text{key}\, x$, go to the left child of x;
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Functional/Recursive Version:

```
search :: Key → BST → Bool
search k Nil = False
search k (Node x l r) =
     (k == x) || if (k < x) then search k l
                            else search k r
```

Imperative/Iterative Version

```
search (t,k):
  x := t
  while x /= Nil and k /= (key x)
    if k < (key x) then x := (left x)
                   else x := (right x)
  return k == (key x)
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This algorithm just returns a Boolean value:
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Exercise: Modify the algorithm for trees containing key-values pairs;
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Exercise: Modify the algorithm for trees containing key-values pairs; if the key is found, it must return the corresponding value.

Complexity: The search algorithm starts at the root and follows a specific path, until it reaches a node that matches the search key or a leaf. The time complexity is $O(h)$ where h is the height of the tree.

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Exercise: Implement the `insert` operation
Hint: Insert in a leaf in the correct position

Minimum and Maximum

The minimum element in a binary search tree is the leftmost one, the maximum is the rightmost

```
minimum :: BST → Maybe Key
minimum Nil = Nothing
minimum (Node x Nil _) = Just x
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In iterative style:

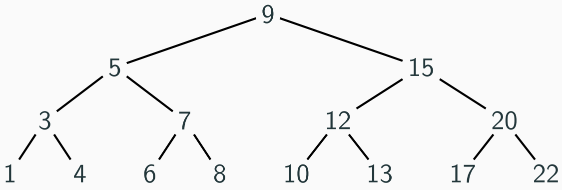
```
minimum (x):
  while (left x) /= Nil
    x := (left x)
  return (key x)
```

The maximum is defined similarly, going right instead of left.

Complexity: $O(h)$ where h is the height of the tree.

Delete

When we delete an element, we search for it, extract it
and replace it with another element that preserves the BST property



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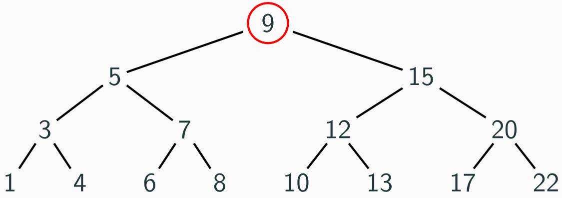

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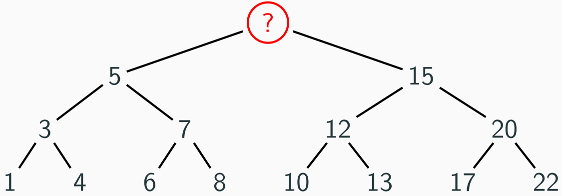

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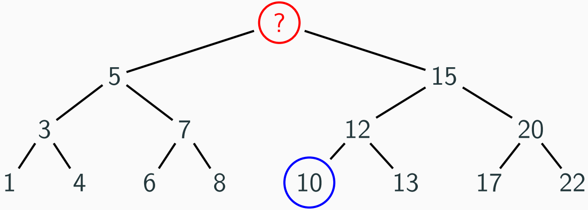

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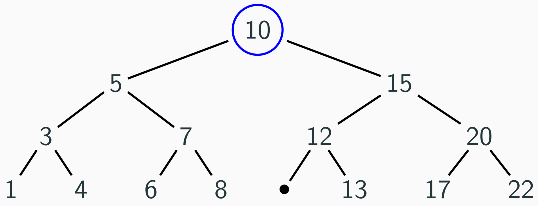

To delete the root 9:

- Remove the root
- Find the minimum of the right child
- Place it at the root

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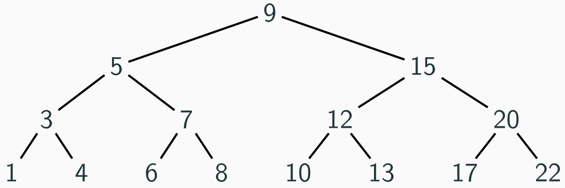
Predecessor and Successor

Extra Operations:

Find the element in the tree that is immediately lower (or higher) than the one at the given pointer.



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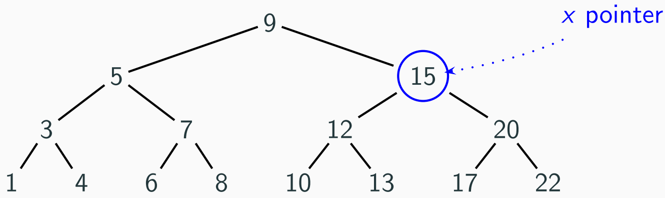

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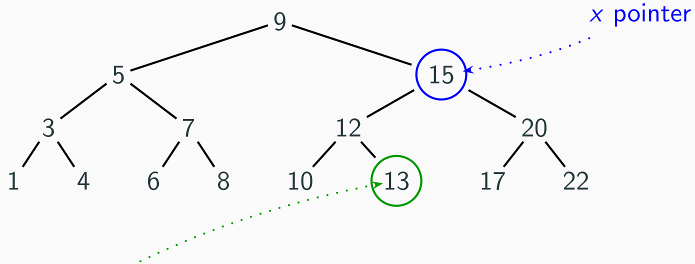

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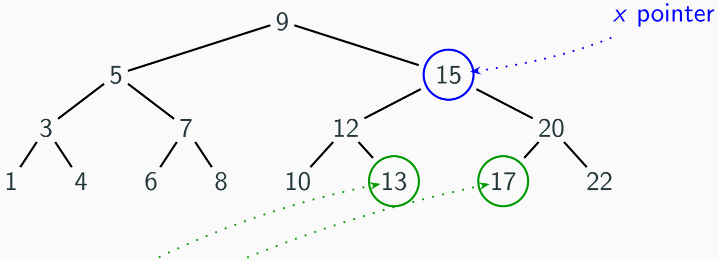

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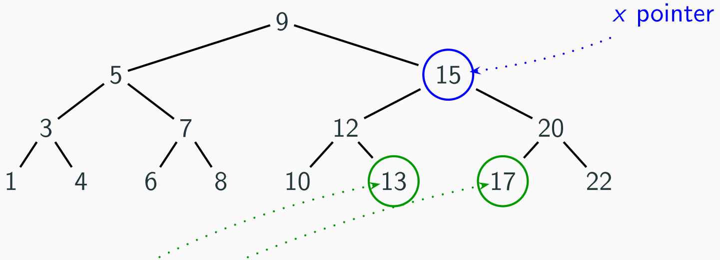

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Exercise: What if the node doesn't have a left child (or a right child)?