Dynamic Programming

Advanced Algorithms and Data Structures - Lecture 5

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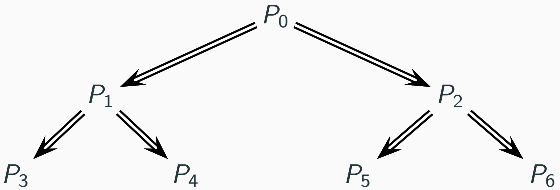
Thursday 29 October 2020

School of Computer Science, University of Nottingham

Repeated Subproblems

- Divide-and-Conquer:
  Split the problem into smaller subproblems - solve them recursively



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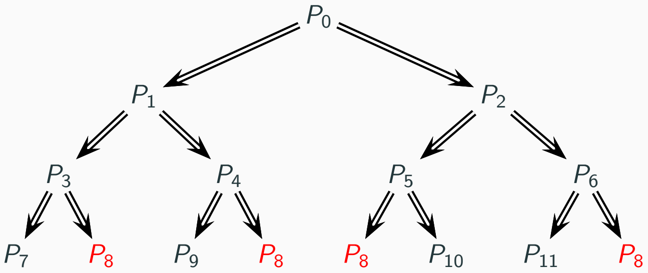

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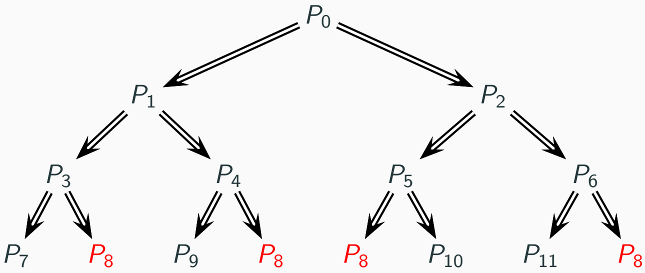

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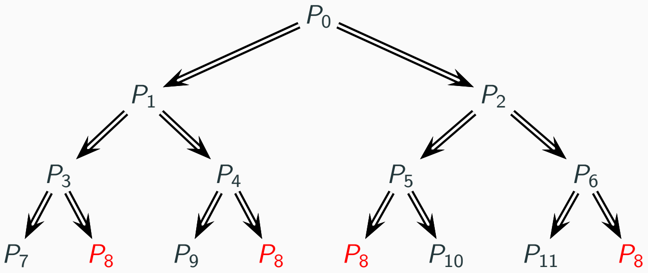
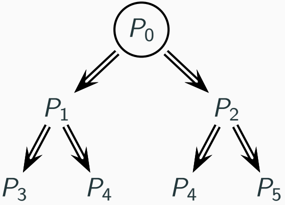


- We may hit the same subproblem in different branches
- Divide-and-Conquer would recompute P_8 four times
- Dynamic programming:
  Remember the solution of P_8 after the first time

Table Building

Dynamic Programming idea:

- Keep a table of already computed subproblems
- Look up a subproblem in the table before recomputing
- New subproblem? Compute the solution and add it to the table



Problem	P_0	P_1	P_2	P_3	P_4	P_5
Solution	?					

The table is a global variable

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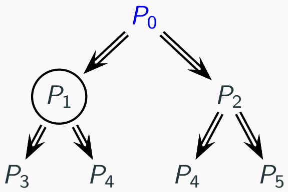

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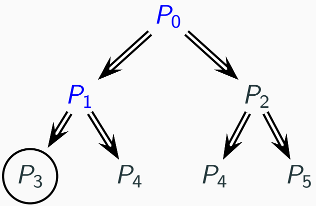

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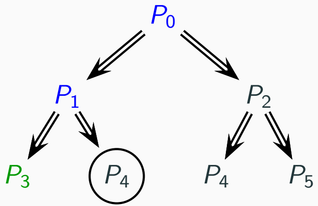

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Solution	?	?		S_3	?	

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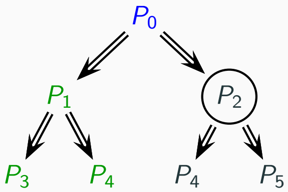

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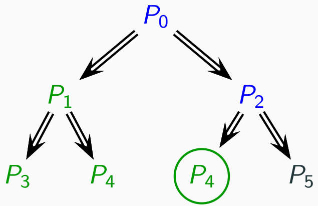

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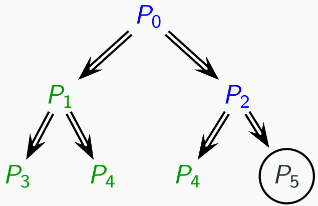

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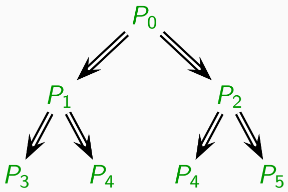

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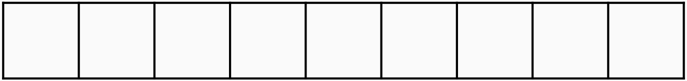
The Rod-Cutting Problem

Cut a rod into pieces maximizing their total price



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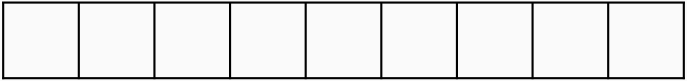
We have a table of prices for pieces of different length

You must cut the rod to maximize the total price of the pieces



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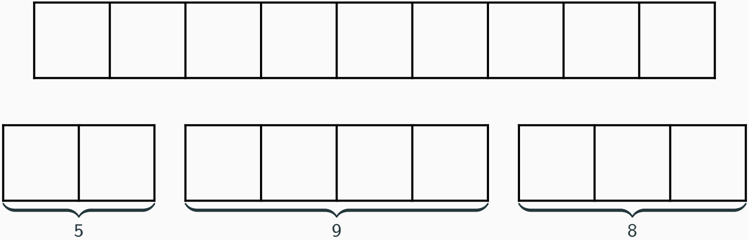

If we split it in $9 = 2 + 4 + 3$, price: $5 + 9 + 8 = 22$

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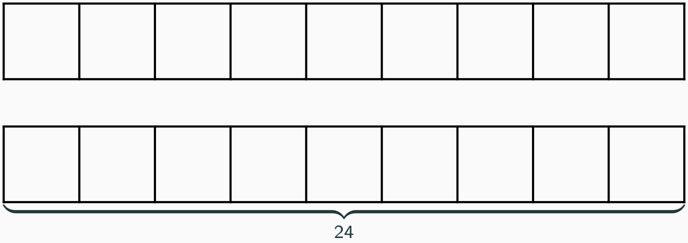

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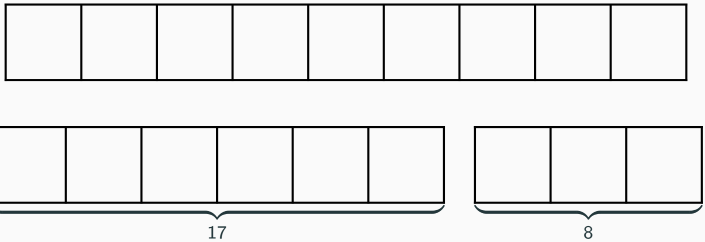

If we split it in $9 = 2 + 4 + 3$, price: $5 + 9 + 8 = 22$

If we don't split it, price: 24

If we split it in $9 = 6 + 3$, price: $17 + 8 = 25$

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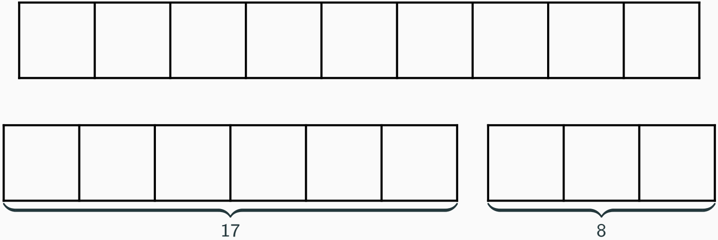

If we split it in $9 = 2 + 4 + 3$, price: $5 + 9 + 8 = 22$

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If we split it in $9 = 6 + 3$, price: $17 + 8 = 25$ (maximum)

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Optimal Cut Equations (1)

We can adopt a divide-and-conquer strategy:

- First do one cut, you get two smaller rods
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Equations expressing the price r_n of an optimal cut of a rod of length n:

$$r_1 = p_1$$
$$r_n = \max\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\}$$

We cut the rod of length n into two rods of length i and $n - i$
in all possible ways (explicitely consider the uncut price p_n)

Optimal Cut Equations (2)

We can improve the algorithm by taking the first cut to be definitive:
The first half will not be further cut,
so we don't need a recursive call for it:

$$r_0 = 0$$
$$r_n = \max_{i=1\dots n}(p_i + r_{n-i})$$

This takes care also of

- r_1 (it automatically gives p_1)
- the uncut option when $i = n$

Observation:
Possible improvement: assume that the first cut is the largest:
Cutting $9 = 3 + 6$ is equivalent to $9 = 6 + 3$
Order of cuts is unimportant: only consider the second one

But we don't follow this path (exercise: try)
We'll look at a better algorithm using Dynamic Programming

Naive Algorithm in Haskell

```
maxCut :: [Int] → Int → Int
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- For longer rods, we create the list of all possible prices
  - Make one cut and recursively compute the price of the optimal cut of the remaining part
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Exercise: Modify it so it returns the cuts (the list of *k*s)

Complexity of Naive Algorithm

The Complexity is Exponential: $T(n) = O(2^n)$
(See IA for the formal derivation)

Problem:
We recompute several times optimal cuts for the same length
Eg when computing `maxCut pr 9`, among the possibilities we have
$9 = 5 + 4$, $9 = 3 + 2 + 4$, $9 = 4 + 1 + 4$ etc
The optimal solution for a rod of length 4 is recomputed each time.

Idea: keep a table with the optimal prices already computed and look up in it before recomputing.

DP Solution: Imperative

We construct a global array/table `bestCut`

that cointains the optimal cut for every length:

`bestCut!!i` = total price of optimal cut for a rod of length `i`

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- Bottom-Up Method: Systematically compute all the values in the table in order: `bestCut!!0`, `bestCut!!1`, ..., `bestCut!!n`; when computing `bestCut!!i`, we already know all the previous values are in the table

Bottom-Up is efficient if we know in advance that we need to compute all the values in the table

DP and lazy evaluation

In Functional Programming:

- Declarative Style: We can just define the table of values, without worrying about the order in which it is computed and when values will be available
- Lazy Evaluation: Entries of the table will be computed when needed and they persist for further calls

```
maxCutD :: [Int] → Int → Int
maxCutD pr n = last bestCut
  where bestCut = 0:[ maximum [ pr!!k + bestCut!!(m-k)
                              | k←[1..m] ]
                    | m ← [1..n] ]
```

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  The optimal solution to an instance of the problem (eg cutting a rod of length n) contains optimal solutions of some subproblems (cutting rods of shorter length)
- Overlapping Subproblems Different branches of the computation of an optimal solution require to compute the same subproblem several times