AADS – Lecture 9

Algorithms in Cryptography and Other Miscellaneous Topics

Intro

On the menu today!

1. Algorithms and Cryptography (*potential exam material*)
    - Symmetric versus asymmetric
    - Meet-in-the-middle
    - RSA, exponentiation by squaring
    - Key sizes

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1. Algorithms and Cryptography (*potential exam material*)
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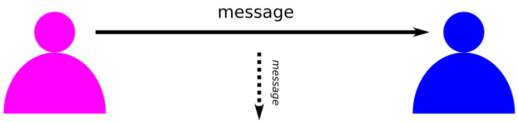
   - RSA, exponentiation by squaring
   - Key sizes
2. Miscellaneous advanced topics in AADS (*not exam material*)
   - Exponential algorithms
   - Approximate algorithms
   - Parallel algorithms
   - Space complexity
   - Average case complexity
   - Worst-case revisited
   - Space complexity

Algorithms in Cryptography

Why talk about crypto?

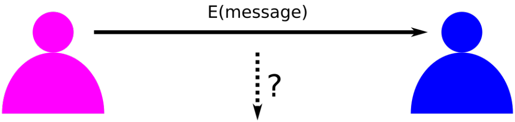
- Loaded with parallels to algorithms!
- Bridges between different disciplines/topics 👍👍👍
- Not a crypto class, different angle
- We simplify things



Problem

- Eavesdroppers can listen in on the channel
- We have to assume channel is insecure / message is "public"

Algorithms in Cryptography – Encryption



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Solution

- The sender *encrypts* the message
- Unintelligible with the correct *key*

Symmetric encryption

- Same key used to encrypt and decrypt
- "Padlock"
- Fast(er)
- In a network of n users, each pair need a distinct key: $n(n+1)/2 = O(n^2)$ keys
- Security (typically) based on established principles
- No proof of security! But not broken after many years...
- AES, 3DES, Blowfish, ChaCha20, ...

Asymmetric encryption

- A *public* key to encrypt, a *private* key to decrypt
- "Mailbox"
- Slow(er)
- Each user has only one (pair of) key(s), no matter how many other agents
- Security (typically) based on hardness assumptions
- Reduction proofs (e.g. reducing to $P = NP$)
- RSA, ElGamal, Pailler, ...

Breaking Symmetric Encryption

- Finding a weakness in the cipher
- A lot of (smart, well-motivated) people have *really* tried
- No proof, but well understood *structures*
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Guessing the Key for a Given Instance

- Keys are k bits (e.g. $k = 128$ for AES)
- All keys equally likely
- All keys produce independent plaintext-ciphertext relationships
- Search through all 2^{128} possibilities
- $T(k) = 2T(k-1)$
- "Exponential algorithm"

Double Encryption

- Encrypt a message twice (with different keys)
- Rationale: larger search space, "patches" cipher if weakness, or if poorly implemented
- Is it useful? Yes and no...

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Meet-in-the-middle Attack

- From a pair x, y such that $y = E_{K_2}(E_{K_1}(x))$
- Compute candidates ciphertexts $E_{K1}(x)$ for all $K_1 \in \{0,1\}^k$
- Insert them in a dictionary
- Compute candidates plaintexts $E_{K_2}^{-1}(y)$ for all $K_2 \in \{0,1\}^k$
- For each, lookup (expected $O(1)$) in dictionary for collision
- Collision found = correct key pair found!

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Solution: Square-and-Multiply

- Exponent in binary form: $e = \overline{e_n \ldots e_1 e_0}$
- From $c = 1$, for each bit in the exponent:
  1. If e_0, $c \leftarrow c \cdot b \bmod N$
  2. In any case, $c \leftarrow c^2 \bmod N$, $e \leftarrow e \gg 1$
- Requires $O(\log e)$ “short” multiplications $\rightarrow$ OK!

S&M makes modular exponentiation feasible on large numbers!

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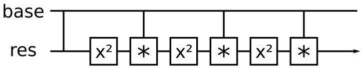

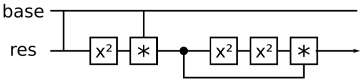

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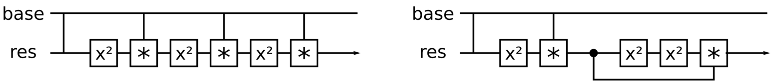

But... finding optimal is hard! Also: only constant factor reduction
Exercise: can DP help here?

RSA

- $N = pq$ with p, q two (large) primes
- Compute $\lambda(N) = lcm(p-1, q-1)$
- Choose e (small) coprime to $\lambda(N)$ and compute $d = e^{-1} \bmod \lambda(N)$; $pk = (e, N)$, $sk = (d, p, q, \lambda(N))$

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Security

- (Assumed) hardness of *RSA problem* (e-th root modulo N)
- (Assumed) hardness of *factoring large composites* (sufficient)

Algorithms in Cryptography – Key Sizes

Symmetric Keys

- Each key in the key space is possible and a priori equally likely
- Brute-force attack takes up to 2^k steps
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Asymmetric Keys

- Not all keys in key space possible
- Some attacks better than brute force, depending on scheme
- For RSA: "general number field sieve" for integer factorization (complexity: $O(\exp((\sqrt[3]{64/9} + o(1))(\ln N)^{\frac{1}{3}}(\ln \ln N)^{\frac{2}{3}})))$
- To remain "equivalent" to 2^{128}, N is 2048 or 4096 bits

Miscellaneous Topics

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Example 1: Traveling Salesman Problem (TSP)

- Given $G = (V, E)$, find $\pi = (v_1, \ldots, v_n)$, a permutation of V, that minimizes $C = \sum_{i=1}^{n} w(v_i, v_{i+1}) + w(v_n, v_1)$
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- Naive algorithm: try all π, compute C for each $\rightarrow O(n \cdot n!)$
- Slightly better exists: $O(n^2 \cdot 2^n)$
- Also, other approaches (branch-and-bound, linear programming)... but nothing "efficient"

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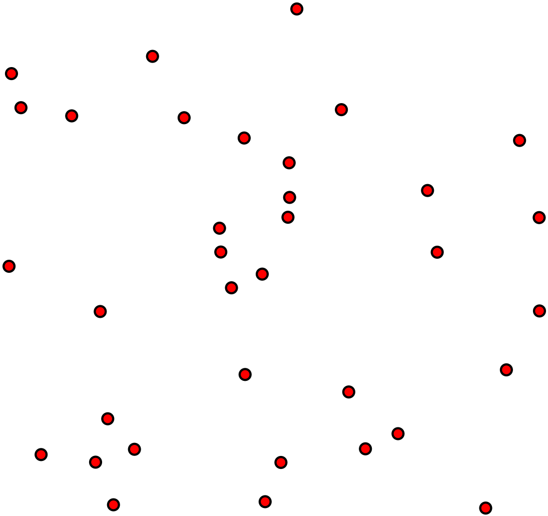

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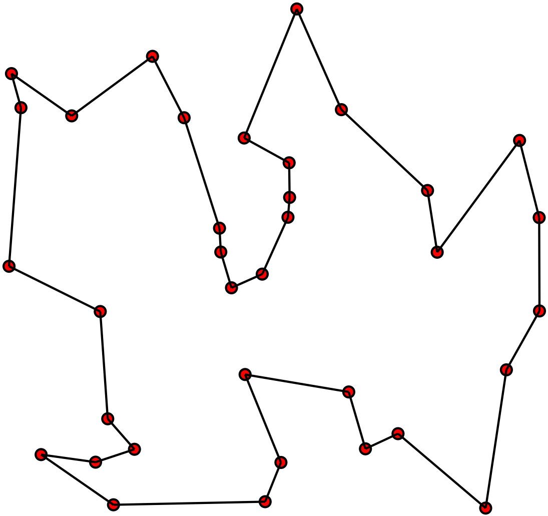

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- **Randomness**, sampling, etc.

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Example: Matrix Multiplication

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

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When Memory Matters!

- Sometimes memory is precious, or even limiting
- Already touched upon once aspect: *implicit data structures*
- In-place versus out-of-place (esp. for *sorting*)
  - e.g. Bubble sort versus Quicksort
- Time-space trade-offs: caching, compression, in crypto, etc.

Other related limitations: $\neq$ kinds of memory, network, etc.

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Examples

- Sorting: all permutations equally likely
- BST Insert: all *possible* BSTs of a given size n
  - Potentially very hard to consider
  - Often: random uniform *sampling* (with care...)

Why Worst Case so Prevalent?

- Easier to work with
- In many cases, *tight* bound
- Sometimes it is precisely what we want to capture
- Real-time and user-facing applications
  - User experience: e.g. web, apps, etc.
  - Safety: e.g. air/train/road traffic signalling
  - Cascading effects: e.g. scheduling